

A COUNTEREXAMPLE TO A HYPOTHESIS OF LIGHT TAIL OF MAXIMUM DISTRIBUTION FOR CONTINUOUS RANDOM PROCESSES WITH LIGHT FINITE-DIMENSIONAL TAILS.

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ABSTRACT.

We construct an example of a continuous centered random process with light tails of finite-dimensional distribution but with heavy tail of maximum distribution.

Key words and phrases: Light and heavy tails of distributions, random process (field), Young-Orlicz function, ordinary and Grand Lebesgue spaces, Orlicz, Lorentz norm and spaces, disjoint sets and functions.

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1 Notations. Statement of problem.

In the article [10], 2008 year, has been formulated the following hypothesis H :

"Let $\theta = \theta(t), t \in T$ be arbitrary separable random field, centered: $\mathbf{E}\theta(t) = 0$, bounded with probability one: $\sup_{t \in T} |\theta(t)| < \infty$ (mod \mathbf{P}), moreover, may be continuous, if the set T is compact metric space relative some distance.

Assume in addition that for some Young (or Young-Orlicz) function $\Phi(\cdot)$ and correspondent Orlicz norm $\|\cdot\|_{Or(\Phi)}$

$$\sup_{t \in T} \|\theta(t)\|_{Or(\Phi)} < \infty. \quad (1)$$

Recall that the Luxemburg norm $\|\xi\|_{Or(\Phi)}$ of a r.v. (measurable function) ξ is defined as follows:

$$\|\xi\|_{Or(\Phi)} = \inf_{k, k > 0} \left\{ \int_{\Omega} \Phi(|\xi(\omega)|/k) \mathbf{P}(d\omega) \leq 1 \right\}.$$

The Young function $\Phi(\cdot)$ is by definition arbitrary even convex continuous strictly increasing on the non-negative right-hand semi-axis such that

$$\Phi(0) = 0, \quad \lim_{u \rightarrow \infty} \Phi(u) = \infty.$$

Let also $\Psi(\cdot)$ be *arbitrary* another Young function such that $\lim_{u \rightarrow \infty} \Psi(u) = \infty$, $\Psi << \Phi$, which denotes by definition

$$\forall \lambda > 0 \Rightarrow \lim_{u \rightarrow \infty} \frac{\Psi(\lambda u)}{\Phi(u)} = 0, \quad (2)$$

see [19], p.16.

Recall that $\Psi \ll \Phi$ implies in particular that the (unit) ball in the space $Or(\Psi)$ is precompact set in the space $Or(\Phi)$.

Open question: there holds (or not)

$$\|\sup_{t \in T} |\theta(t)|\|_{Or(\Psi)} < \infty. \quad (3)$$

The conclusion (3) is true for the centered (separable) Gaussian fields [1], if the field $\theta(\cdot)$ satisfies the so-called entropy or generic chaining condition [11], [12], [11], [8], [21], [22], [23]; in the case when $\theta(\cdot)$ belongs to the domain of attraction of Law of Iterated Logarithm [13] etc.

Notice that if the field $\theta(t)$ is continuous (mod \mathbf{P}) and satisfies the condition (1), then *there exists* an Young function $\Psi(\cdot)$, $\Psi(\cdot) \ll \Phi(\cdot)$ for which the inequality holds, see [10].

The condition of a view $\|\xi\|_{Or(\Psi)} < \infty$ described the tail behavior for the distribution of the random variable ξ . Another approach which was used in the monograph M.Ledoux and M.Talagrand [8], p. 309-317 is related in fact with generalized Lorentz (more exactly, Lorentz -Zygmund) norm $\|\xi\|_{L(v)}$:

$$\|\xi\|_{L(v)} \stackrel{def}{=} \sup_{A: \mathbf{P}(A) > 0} \left[\frac{1}{v(\mathbf{P}(A))} \cdot \int_A |\xi(\omega)| \mathbf{P}(d\omega) \right].$$

Here $v = v(z)$, $z \in (0, 1]$ is continuous monotonically increasing function such that $v(0) = 0$.

Notice that in all this cases the inequality (3) is true with replacing the function Ψ on the function Φ .

Our target is to give a negative answer on this hypothesis, by means of construction of correspondent counterexample.

We consider in this short article also the case of unbounded measure \mathbf{P} .

The detail investigation of the theory of Orlicz's spaces included the case unbounded measure see in the monographs [19], [20].

The notion of separability and entropy or generic chaining condition for the conclusion (3) in this case see in [15].

SEVERAL NOTATIONS AND DEFINITIONS.

A. A triplet $(\Omega, \mathcal{B}, \mathbf{P})$, where $\Omega = \{\omega\} = \{x\}$ is arbitrary set, \mathcal{B} is non-trivial sigma-algebra subsets Ω and \mathbf{P} is non-zero non-negative completely additive measure defined on the \mathcal{B} is called a probabilistic space, even in the case when $\mathbf{P}(\Omega) = \infty$. We denote as usually for the random variable ξ (r.v.) (i.e. measurable function $\Omega \rightarrow R$)

$$|\xi|_p = [\mathbf{E}|\xi|^p]^{1/p} = \left[\int_{\Omega} |\xi(\omega)|^p \mathbf{P}(d\omega) \right]^{1/p}, \quad p \geq 1;$$

$$L_p = \{\xi, |\xi|_p < \infty\}.$$

B. The so-called Grand Lebesgue Space $G\psi$ with norm $\|\cdot\|_{G\psi}$ is defined (in this article) as follows:

$$G\psi = \{\xi, \|\xi\|_{G\psi} < \infty\}, \quad \|\xi\|_{G\psi} \stackrel{\text{def}}{=} \sup_{p \geq 1} \left[\frac{|\xi|_p}{\psi(p)} \right]. \quad (4)$$

Here $\psi = \psi(p)$ is some continuous increasing function such that $\lim_{p \rightarrow \infty} \psi(p) = \infty$.

The detail investigation of this spaces (and more general spaces) see in [9], [16]. See also [2], [3], [5], [6], [7] etc.

An important for us fact about considered here spaces is proved in [14]: they coincide with some exponential Orlicz's spaces $Or(\Phi_\psi)$. For instance, if $\mathbf{P}(\Omega) = 1$ and $\psi(p) = \psi_{1/2}(p) := \sqrt{p}$, then the space $G\psi_{1/2}$ consists on all the subgaussian (non-centered, in general case) r.v. $Or(\Phi_{\psi_{1/2}})$ for which $\Phi_{\psi_{1/2}}(u) = \exp(u^2/2) - 1$.

The Gaussian distributed r.v. η belongs to this space. Another example: let $\Omega = (0, 1)$ with usually Lebesgue measure and

$$f_{1/2}(\omega) = \sqrt{|\log \omega|}, \quad \omega > 0; \quad f_{1/2}(0) = 0.$$

It is easy to calculate using Stirling's formula for the Gamma function:

$$|f_{1/2}|_p \asymp \sqrt{p}, \quad p \in (1, \infty).$$

The tail behavior:

$$\mathbf{P}(f_{1/2} > u) = \exp(-u^2).$$

The case when in (4) supremum is calculated over *finite* interval is investigated in [9], [16], [17]:

$$G_b\psi = \{\xi, \|\xi\|_{G_b\psi} < \infty\}, \quad \|\xi\|_{G_b\psi} \stackrel{\text{def}}{=} \sup_{1 \leq p < b} \left[\frac{|\xi|_p}{\psi(p)} \right], \quad b = \text{const} > 1, \quad (5)$$

but in (5) $\psi = \psi(p)$ is continuous function in the semi-open interval $1 \leq p < b$ such that $\lim_{p \downarrow b} \psi(p) = \infty$.

An used further example:

$$\psi^{(\beta, b)}(p) = (b - p)^{-\beta}, \quad 1 \leq p < b, \beta = \text{const} > 0; \quad G_{\beta, b}(p) := G_b\psi^{(\beta, b)}(p).$$

C. Recall that sets $A_1, A_2, A_i \in \mathcal{B}$ are disjoint, if $A_1 \cap A_2 = \emptyset$. The sequence of a functions $\{h_n\}, n = 1, 2, 3, \dots$ is said to be *disjoint*, or more exactly *pairwise disjoint*, if

$$\forall i, j; i \neq j \Rightarrow h_i \cdot h_j = 0. \quad (6)$$

If the sequence of a functions $\{h_n\}$ is pairwise disjoint, then

$$|\sum_n h_n|_p^p = \sum_n |h_n|_p^p, \quad \sup_n |h_n(x)| = \sum_n |h_n(x)|. \quad (7)$$

D. We denote as ordinary for any measurable set A , $A \in \mathcal{B}$ its indicator function by $I(A) = I_A(\omega)$.

2 Main result.

Theorem. The proposition of hypothesis H is not true. Indeed, there exist:

A. A centered continuous in the $Or(\Phi)$ sense and with probability one random process (field) $\theta = \theta(t) = \theta(t, \omega)$ defined aside from the probabilistic space on some *compact* metric space $(T, d) = (\{t\}, d)$, where d is the non-trivial distance in the set T , such that for some Young function $\Phi = \Phi(u)$ the condition (1) is satisfied.

B. An Young function $\Psi = \Psi(u)$ for which $\lim_{u \rightarrow \infty} \Psi(u) = \infty$, $\Psi \ll \Phi$ but

C.

$$\|\sup_{t \in T} |\theta(t)|\| Or(\Psi) = \infty. \quad (8)$$

Proof.

1. In the sequel we choose as a capacity metric space (T, d) the set of positive integer numbers with infinite point which we denote ∞ :

$$T = \{1, 2, 3, \dots, \infty\}.$$

The distance d is defined as follows:

$$d(i, j) = \left| \frac{1}{i} - \frac{1}{j} \right|, \quad i, j < \infty; \quad d(i, \infty) = d(\infty, i) = \frac{1}{i}, \quad i < \infty; \quad (9)$$

and obviously $d(\infty, \infty) = 0$.

The pair (T, d) is compact (closed) metric space and the set T has unique limit point $t_0 = \infty$. For instance, $\lim_{n \rightarrow \infty} d(n, \infty) = 0$.

2. We consider first of all the case $\mathbf{P}(\Omega) = \infty$. Introduce as an example the following triplet: $\Omega = \{x\} = R_+$, \mathcal{B} is Borelian sigma-algebra and \mathbf{P} is Lebesgue measure: $\mathbf{P}(dx) = dx$.

Let us define a numerical sequences $c(n) = \log^{-3}(n+3)$, $n \in T$ and a sequence of functions

$$g_n(x) = c(n) \cdot I_{(n, n+1)}(x) \cdot f_{1/2}(x-n), \quad g_\infty(x) = 0, \quad (10)$$

$$g(x) = \sum_{n=1}^{\infty} g_n(x). \quad (11)$$

Note that the functions g_n are disjoint and following $\sup_n |g_n(x)| < \infty$ almost surely.

We calculate using the relations (7):

$$|g_n|_p = c(n) \psi_{1/2}(p), \quad (12)$$

therefore $\|g_n - g_\infty\|_{G\psi_{1/2}} = \|g_n\|_{G\psi_{1/2}} \rightarrow 0$ as $n \rightarrow \infty$ and moreover $g_n \rightarrow 0$ almost everywhere. Indeed, let ϵ be arbitrary positive number. We get:

$$\sum_n \mathbf{P}(|g_n| > \epsilon) \leq \sum_n \exp(-C\epsilon^2 \log^6(n)) < \infty.$$

Our conclusion follows from the lemma of Borel- Cantelli, which is true even for unbounded measure.

So, the process $\theta(t) = g_n$, $n = t$ satisfies the condition (1) relative the Young function $\Phi_{\psi_{1/2}}(u)$.

Let now p be arbitrary number, $1 \leq p < \infty$. We have:

$$\left| \sup_n |g_n| \right|_p^p = \sum_n |g_n|_p^p = \sum_n c^p(n) \psi_{1/2}^p(p) = \infty. \quad (13)$$

So, we can choose in the capacity of the Young function $\Psi = \Psi(u)$ the function $\Psi(u) = |u|^p$.

In order to obtain the centered needed process $\theta(t)$ with at the same properties, we consider the sequence $\tilde{g}_n(x) = \epsilon(n) \cdot g_n(x)$, where $\{\epsilon(n)\}$ is a Rademacher sequence independent on the $\{g_n\}$:

$$\mathbf{P}(\epsilon(n) = 1) = \mathbf{P}(\epsilon(n) = -1) = 1/2;$$

then $|\tilde{g}_n(x)| = |g_n(x)|$, $\|\tilde{g}_n\|_p = \|g_n\|_p$ and the sequence $\{\tilde{g}_n\}$ is also pairwise disjoint (Rademacher's symmetrization).

This completes the proof of our theorem, but only in the case $\mathbf{P}(\Omega) = \infty$.

3. The case $\mathbf{P}(\Omega) = 1$ is more complicated. We choose $\Omega = (0, 1)$ with Lebesgue measure and define

$$a(n) = 1 - 0.5n^{-\alpha}, \quad \alpha = \text{const} \in (0, 1); \quad p_0 = \text{const} > 1;$$

$$\Delta(n) = a(n+1) - a(n) \sim_{n \rightarrow \infty} C_\alpha n^{-\alpha-1}; \quad c(n) = n^{\alpha/p_0}. \quad (14)$$

We introduce as before the following positive random process $\theta(t) = g_n$, $n = t$, $t, n \in T$, $\Omega = \{x\}$,

$$g_n(x) = c(n) f_{1/2} \left(\frac{x - a(n)}{\Delta(n)} \right) I_{(a(n), a(n+1))}(x), \quad x \in \Omega, \quad g_\infty(x) = 0; \quad (15)$$

$$g(x) = \sum_n g_n(x) = \sum_n n^{\alpha/p_0} f_{1/2} \left(\frac{x - a(n)}{\Delta(n)} \right) I_{(a(n), a(n+1))}(x). \quad (16)$$

Note that the sequence of r.v. $\{g_n(x)\}$ is again non-negative and disjoint, therefore

$$\sup_n g_n(x) = \sum_n g_n(x) = g(x), \quad \left| \sup_n g_n \right|_p^p = \sum_n |g_n|_p^p. \quad (17)$$

Remark 1. In this pilcrow $c(n) \rightarrow \infty$, in contradiction to the case $\mathbf{P}(\Omega) = \infty$.

Let now $p \in [1, p_0)$; in what follows we presume $p \rightarrow p_0 - 0$. We calculate consequently, as long as the function $\psi_{1/2}(p)$ is bounded: $\psi_{1/2}(p) \leq \sqrt{p_0}$:

$$|g_n|_p^p = n^{\alpha p/p_0} \Delta(n) \sim C_\alpha n^{-1-\alpha(1-p/p_0)}, \quad (18)$$

therefore $|g_n - g_\infty|_{p_0} \rightarrow 0$, $n \rightarrow \infty$. The L_{p_0} and with probability one continuity of the process g_n , $n \in T$ it follows from (18) and the main result of an article of G.Pizier [18].

Further,

$$\begin{aligned} |g|_p^p / C_\alpha &= \sum_n n^{-1-\alpha(1-p/p_0)} \sim C_1(\alpha) (p_0 - p)^{-1}; \\ |g|_p &\sim C_2(\alpha) (p_0 - p)^{-1/p} \sim C_3(\alpha) (p_0 - p)^{-1/p_0}. \end{aligned} \quad (19)$$

The equality (19) implies on the language of $G\psi$ spaces that

$$\| \sup_n |g_n| \| G_{1/p_0, p_0} < \infty, \quad (20)$$

and that the relation (20) is exact.

It follows from the equalities (19) and (20) that the tail function for the r.v. $g = g(x)$, i.e. the function $G_g(z) := \mathbf{P}(|g(x)| \geq z)$, $z > 0$ obeys the following asymptotical as $z \rightarrow \infty$ expression:

$$G_g(z) \sim C_4(\alpha) z^{-p_0}, \quad (21)$$

i.e. at the same asymptotic as for the r.v. $\eta = \eta(x) = C_5(\alpha) x^{-1/p_0}$.

We offer in the capacity of the Young function $\Phi = \Phi(u)$ the Young-Orlicz function for the L_{p_0} space: $\Phi(u) = |u|^{p_0}$.

Let us consider the following Young function:

$$\Psi(u) = |u|^{p_0} (\log(e + |u|))^{-1/2}. \quad (21)$$

By virtue of the condition $\mathbf{P}(\Omega) = 1$ we deduce $\Psi \ll \Phi$.

Since the Orlicz spaces are rearrangement invariant,

$$\|g\| Or(\Psi) < \infty \Leftrightarrow \|\eta\| Or(\Psi) < \infty.$$

As long as the function $\Psi = \Psi(u)$ satisfies the Δ_2 condition,

$$\|\eta\| Or(\Psi) < \infty \Leftrightarrow \int_\Omega \Psi(|\eta(x)|) dx < \infty.$$

But

$$\int_\Omega \Psi(|\eta(x)|) dx \geq C_6 + C_7 \int_0^{1/2} x^{-1} |\log x|^{-1/2} dx = \infty. \quad (22)$$

Thus,

$$\| \sup_n |g_n| \| G\Psi = \| |g| \| G\Psi = \infty. \quad (23)$$

It remains to use the known Rademacher's symmetrization method in order to obtain the centered process $\theta(t)$.

Remark 2. The constructed process $\theta(t)$ give us a new example of centered continuous random process with light tails of finite-dimensional distribution, but for which entropy and generic chains series divergent.

Remark 3. The proposition of our theorem remains true with at the same (counter) example if we use instead Orlicz space the generalized Lorentz (Lorentz -Zygmund) space $L(v)$ or the space $K = K(h) = \{\tau = \tau(\omega)\}$ with quasinorm

$$\|\tau\|_K = \sup_{z>0} [G_\tau(z)/h(z)].$$

Here $h = h(z)$, $z \in (0, \infty)$ is continuous monotonically decreasing function such that $v(0+) = \mathbf{P}(\Omega)$.

It is important for us only that the tail of distribution for $\sup_t \theta(t)$ is essentially greatest in comparison with $1/\Phi(u/C)$ for arbitrary constant $C > 0$.

Remark 4. The proposition of our theorem remains true if we use instead the space of continuous function $C(T, d)$ arbitrary separable Banach space.

References

- [1] FERNIQUE X. (1975). Regularite des trajectoires des fonction aleatoires gaussiennes. Ecole de Probablite de Saint-Flour, IV, Lecture Notes in Mathematic. 480, **1**, 96, Springer Verlag, Berlin.
- [2] CAPONE C., FIORENZA A., KRBEC M. On the Extrapolation Blowups in the L_p Scale. Collectanea Mathematica, **48**, 2, (1998), 71 - 88.
- [3] FIORENZA A. Duality and reflexivity in grand Lebesgue spaces. Collectanea Mathematica (electronic version), **51**, 2, (2000), 131 - 148.
- [4] FIORENZA A., AND KARADZHOV G.E. Grand and small Lebesgue spaces and their analogs. Consiglio Nazionale Delle Ricerche, Istituto per le Applicazioni del Calcolo Mauro Picine, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).
- [5] IWANIEC T., AND SBORDONE C. On the integrability of the Jacobian under minimal hypotheses. Arch. Rat.Mech. Anal., 119, (1992), 129 - 143.
- [6] IWANIEC T., P. KOSKELA P., AND ONNINEN J. Mapping of finite distortion: Monotonicity and Continuity. Invent. Math. 144 (2001), 507 - 531.
- [7] KOZACHENKO YU. V., OSTROVSKY E.I. (1985). The Banach Spaces of random Variables of subgaussian type. *Theory of Probab. and Math. Stat.* (in Russian). Kiev, KSU, **32**, 43 - 57.
- [8] LEDOUX M., TALAGRAND M. (1991) Probability in Banach Spaces. Springer, Berlin, MR 1102015.

- [9] LIFLYAND E., OSTROVSKY E., SIROTA L. Structural Properties of Bilateral Grand Lebesgue Spaces.
- [10] OSTROVSKY E. Support of Borelian Measures in separable Banach Spaces. arXiv:0808.03248v1 [math.FA] 24 Aug 2008.
- [11] OSTROVSKY E., ROGOVER E. Exact exponential Bounds for the random field Maximum Distribution via the majorizing Measures (Generic Chaining). arXiv:0802.0349v1 [math.PR] 4 Feb 2008.
- [12] OSTROVSKY E.I. (1999). Exponential estimations for random Fields and its applications (in Russian). Moscow - Obninsk, OINPE.
- [13] OSTROVSKY E.I. (1994.) Exponential Bounds in the Law of Iterated Logarithm in Banach Space. Math. Notes, **56**, 5, p. 98 - 107.
- [14] OSTROVSKY E., SIROTA L. Fouier Transforms in exponential rearrangement invariant Spaces. arXiv:040639v1 [math.FA], 20 Jun 2004.
- [15] OSTROVSKY E., ROGOVER E. Naximal Inequalities in bilateral Grand Lebesgur Spaces over unbounded Measure. arXiv:0808v1 [math.FA], 24 Aug 2008.
- [16] OSTROVSKY E. AND SIROTA L. Moment Banach spaces: theory and applications. HIAT Journal of Science and Engineering, **C**, Volume 4, Issues 1 - 2, pp. 233 - 262, (2007).
- [17] OSTROVSKY E. AND SIROTA L. Moment and Tail Inequalities for polynomial Martingales. The case of heavy tails. arXiv: 1112.2768v1 [math.PR] 13 Dez 2011.
- [18] PIZIER G. Condition d'entropic assupant la continuite de certains processus et applications a lanalyse harmonique. Seminaire d'analyse fonctionnelle. (1980), Exp.13, p. 23-34.
- [19] RAO M.M., REN Z.D. Theory of Orlicz Spaces. Marcel Dekker Inc., 1991. New York, Basel, Hong Kong.
- [20] RAO M.M., REN Z.D. Applications of Orlicz Spaces. Marcel Dekker Inc., 2002. New York, Basel, Hong Kong.
- [21] TALAGRAND M. (1996). Majorizing measure: The generic chaining. *Ann. Probab.* **24** 1049 - 1103. MR1825156
- [22] TALAGRAND M. (2001). Majorizing Measures without Measures. *Ann. Probab.* **29**, 411-417. MR1825156
- [23] TALAGRAND M. (2005). The Generic Chaining. Upper and Lower Bounds of Stochastic Processes. Springer, Berlin. MR2133757.
- [24] TALAGRAND M.(1990). Sample boundedness of stochastic processes under increment conditions. *Ann. Probab.* **18**, 1 - 49.